[11:00] Connection between convolution and probability distributions

Example: 6-sided die (discrete uniform distribution)

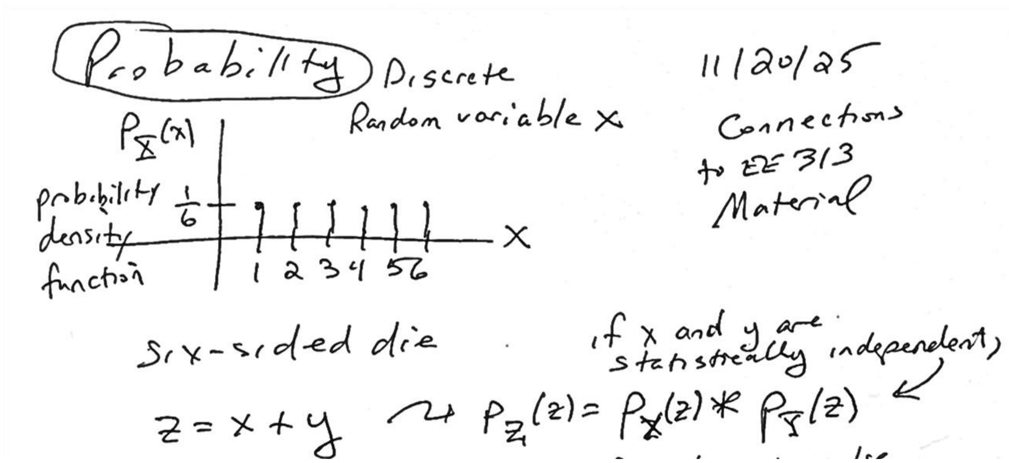
Let X and Y be random variables representing the outcome of rolling two six-sided dice

$$P_X(x) = \begin{cases} 1/6 & x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$$

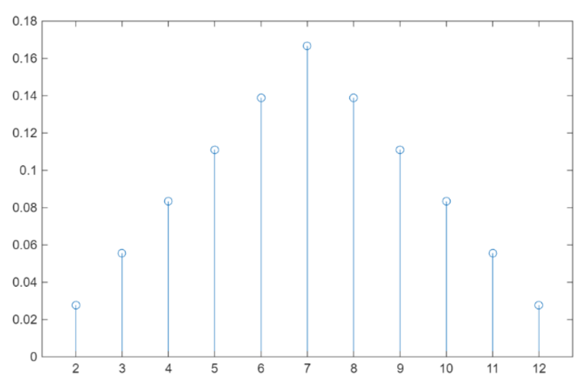
$$P_Y(y) = \begin{cases} 1/6 & y \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$$

If X and Y are statistically independent, the probability density function of the random variable of $Z = X + Y$ is the convolution

$$p_Z(z) = p_X(z) * p_Y(z)$$



px = 1/6 * [1 1 1 1 1 1];
py = 1/6 * [1 1 1 1 1 1];
pz = conv(px, py);
n = 2:12;
stem(n, pz);



[11:10] LTI Systems and continuous time convolution

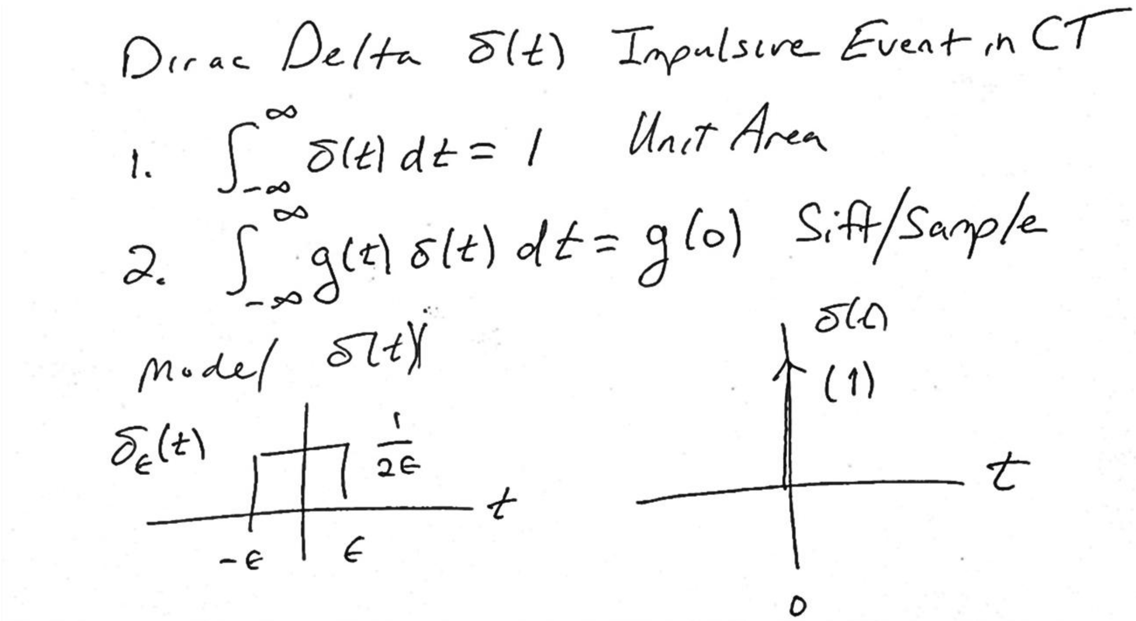
Review from Lecture 12: The Dirac delta is defined by two properties:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad \text{(Unit area)}$$

$$\int_{-\infty}^{\infty} g(t)\delta(t)\,dt = g(0) \qquad \text{(Sifting property)}$$

Using a substitution of variables, the sifting property for a delayed Dirac delta becomes

$$\int_{-\infty}^{\infty} g(t)\delta(t-T)\,dt = g(T) \qquad \text{(Sifting property for } \delta(t-T)\text{)}$$



For a continuous-time LTI system, the response to an impulse (Dirac delta) uniquely characterizes the system. The output is the convolution of the input and the impulse response:

$$\underbrace{y(t)}_{\substack{\text{output of}\\ \text{LTI system}}} = \overbrace{\underbrace{h(t)}_{\substack{\text{impulse}\\ \text{response}}} * \underbrace{x(t)}_{\text{input}}}^{\text{convolution}} = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Example (gain) $\boldsymbol{y(t) = a_0 x(t), a_0 \neq 0}$

$$\mathcal{S}\{ax(t)\} = a_0\big(ax(t)\big), \qquad a\mathcal{S}\{x(t)\} = a\big(a_0x(t)\big)$$

$$\mathcal{S}\{ax(t)\} = a\mathcal{S}\{x(t)\} \qquad \text{(homogeneity)}$$

$$\mathcal{S}\{x_1(t) + x_2(t)\} = a_0(x_1(t) + x_2(t)), \qquad \mathcal{S}\{x_1(t)\} + \mathcal{S}\{x_2(t)\} = a_0x_1(t) + a_0x_2(t)$$

$$\{x_1(t) + x_2(t)\} = \mathcal{S}\{x_1(t)\} + \mathcal{S}\{x_2(t)\} \qquad \text{(additivity)}$$

$$\mathcal{S}\{x(t-t_0)\} = a_0x(t-t_0), \qquad \mathcal{S}\{x(\tau)\}\big|_{\tau=t-t_0} = a_0x(\tau)\big|_{\tau=t-t_0}$$

$$\mathcal{S}\{x(t-t_0)\} = \mathcal{S}\{x(\tau)\}\big|_{\tau=t-t_0} \qquad \text{(time invariance)}$$

If the input is an impulse $\delta(t)$, then the output is $h(t) = a_0\delta(t)$

$$y(t) = h(t) * x(t) = \underbrace{\int_{\tau=-\infty}^{\infty} a_0\delta(\tau)x(t-\tau)\,d\tau = a_0x(t)}_{\text{sifting property}}$$

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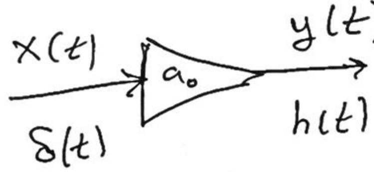
Gain Block
Slide 13-4



$y(t) = a_0 x(t)$

$h(t) = a_0 \delta(t)$

a_0 is a non-zero constant

Linearity: Homogeneity.

input $a\,x(t)$ output $y_{scaled}(t) = a_0(a\,x(t))$

$y_{scaled}(t) = a(a_0 x(t)) = a\,y(t)$

Linearity: Additivity

input $x_1(t) + x_2(t)$

output $y_{additive}(t) = a_0(x_1(t) + x_2(t))$

$= a_0 x_1(t) + a_0 x_2(t)$

$= y_1(t) + y_2(t)$

Time-Invariance:

input $x(t - t_0)$

output $y_{shifted}(t) = a_0(x(t-t_0))$

$= a_0 x(t-t_0)$

$= a_0 y(t)\big|_{t=t-t_0}$

$= a_0 y(t-t_0)$

All pointwise systems are time-invariant

Example (delay) $y(t) = x(t-T)$, T is the delay in seconds

If the system were a long wire, the length of the wire is $L = \alpha\, c\, T$ where $c = 3 \times 10^8$ m/s and the speed of electrons along the wire is $\alpha\, c$. Along the wire at each point in space, there is a voltage and a current.

Initial Conditions. If we start observing at time $t = 0$, we have to wait T seconds until the input signal arrives on the output. In the meantime, we observe initial conditions present along the long wire connecting the input to the output in the system:

$$y(t) = \begin{cases} \text{initial conditions} & 0 \le t < T \\ x(T) & t \ge T \end{cases}$$

In lab, we enforce initial conditions to be zero by grounding the wire for T seconds or to be near zero by applying a very low voltage for T seconds.

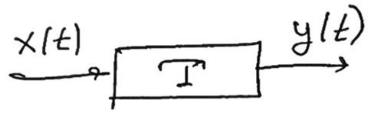
When observing for all time, we don't have to worry about the initial conditions.

Impulse Response. Input $\delta(t)$ for $-\infty < t < \infty$; impulse response is $h(t) = \delta(t - T)$.

$$y(t) = x(t) * h(t) = \underbrace{\int_{\tau=-\infty}^{\infty} \delta(\tau - T)\, x(t - \tau)\, d\tau = x(t - T)}_{\text{sifting property}}$$

Ideal Delay 11/20/25

Slide 13-4



- T is the delay in seconds.
- If the system were a long wire, the ~~d~~ length of the wire is $d = cT$ where $c = 3 \times 10^8$ m/s and T is in seconds.
- Along the wire ~~at~~ at each point in space, there's a voltage and a current
- If we start observing at ~~t~~ $t=0$,

$$y(t) = \begin{cases} \text{init. cond.} & 0 \le t < T \\ x(t-T) & t \ge T \end{cases}$$

- Initial conditions are a buffer of signal lasting T seconds, $v(t)$.
- Initial conditions must be zero as a necessary condition for linearity to hold.

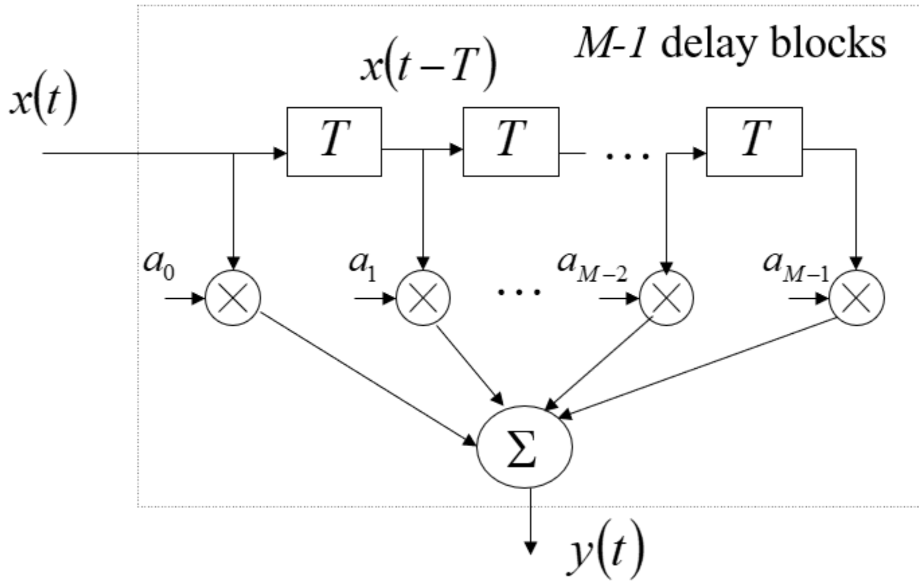
[11:50] Integrator and tapped delay line

Example (integrator) $y(t) = \int_{\tau=-\infty}^{t} x(t)\, d\tau$

The response to an impulse $\delta(t)$ is $h(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, \underbrace{u(t-\tau)}_{\substack{0 \text{ if } t<\tau \\ 1 \text{ if } t\ge\tau}}\, d\tau = \int_{\tau=-\infty}^{t} x(t)\, d\tau$$

Example (tapped delay line) $y(t) = a_0\delta(t) + a_1\delta(t-T) + \cdots + a_{M-1}\delta(t-(M-1)T)$



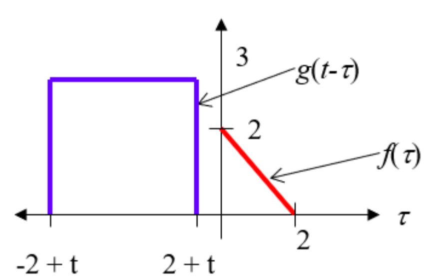
When the input is an impulse $\delta(t)$, the output is $h(t) = \sum_{m=0}^{M-1} a_m \delta(t - mT)$

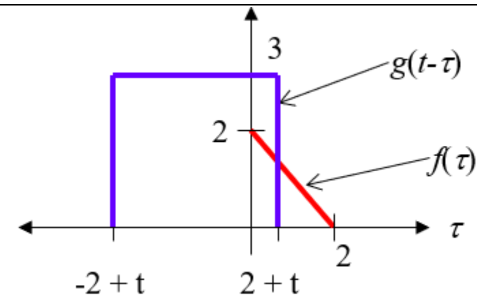
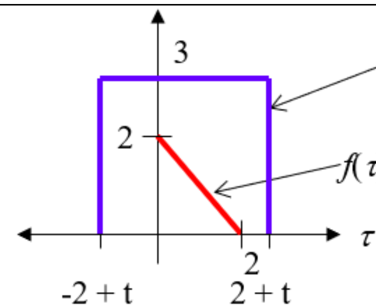
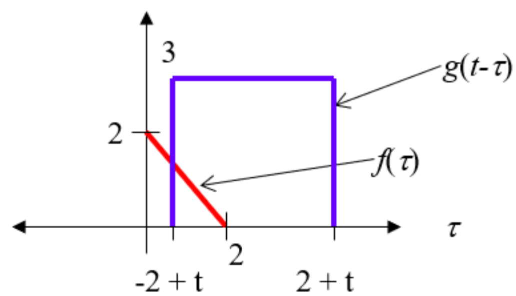
[12:00] Continuous-time convolution (graphical method)

Sometimes called “flip and slide” convolution

$$f_1(t) * f_2(t) = \underbrace{\int_{\tau=-\infty}^{\infty}}_{\text{area}} \overbrace{f_1(\tau)\, f_2(\underbrace{t}_{\substack{\text{shift} \\ \text{by t}}} \underbrace{-\tau}_{\text{flip}})}^{\text{product}}\, d\tau$$

When convolving two finite-length signals, we can divide the output into 5 cases:

No overlap $t < -2$	

Partial overlap $-2 \le t < 0$	
Full overlap $0 \le t < 2$	
Partial overlap $2 \le t < 4$	
No overlap $t \ge 4$	